Math 210B Lecture 16 Notes

Daniel Raban

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1 Change of Basis, Characteristic Polynomials, Trace, and Localization of Modules

1.1 Change of basis

Last time, we discussed $Q_{B,B'}$, the change of basis matrix from $B \to B'$.

Remark 1.1. From the definition, we can see $Q_{B,B'}^{-1} = Q_{B',B}$.

Theorem 1.1 (change of basis)**.** *Let $T : V \to W$ be a homomorphism of free R-modules of finite rank. Let B and B' be ordered basis of V, and let C and C' be ordered bases of W. If A represents T with respect to B and C, then $Q_{C',C'}AW_{B,B'}$ represents T with respect to B' and C'.*

Proof. Note that
$$\varphi_{C'}^{-1}T\varphi_{B'} = (\varphi_{C'}^{-1}\varphi_C)(\varphi_C^{-1}T\varphi_{B'})(\varphi_B\varphi_B^{-1}).$$
The left hand side represents T with respect to B' and C'. The right hand side terms are represented by $Q_{C,C'}^{-1}$, A, and $Q_{B,B'}$, respectively. $\square$

Definition 1.1. A and A' in $M_n(R)$ are **similar** if there exists some $Q \in \mathrm{GL}_n(R)$ such that $A' = Q^{-1}AQ$.

Definition 1.2. A is **diagonalizable** if it is similar to a diagonal matrix.

1.2 Characteristic polynomials and trace

Now suppose that $R = F$ is a field.

Definition 1.3. The **characteristic polynomial** $c_T \in F[x]$ of an F-linear trnasformation $T : V \to V$ of vector spaces is $\det(x \,\mathrm{id} - T)$.

Here, $x \,\mathrm{id} - T : F[x] \otimes_F V \to F[x] \otimes_F V$, where $x \,\mathrm{id} - T$ is really $x \otimes \mathrm{id} - \mathrm{id} \otimes T$. This is a map of free modules of finite rank. Similarly, we have $c_A(x) \in F[x]$ for $A \in M_n(F)$, where $c_A(x) = \det(xI - A)$, and $xI - A \in M_n(F[x])$.

Remark 1.2. $c_T(x) = c_A(x)$ for A representing T with respect to some basis B. This is independent of the basis B. Let $H = Q^{-1}AQ$. Then

$$\begin{aligned} c_H(x) &= \det(xI - Q^{-1}AQ) = \det(Q^{-1}(xI - a)Q) \\ &= \det(Q)^{-1}\det(xI - A)\det(Q) = \det(xI - A) \\ &= c_A(x). \end{aligned}$$

Remark 1.3. If $T(v) = \lambda v$ for $v \in V, \lambda \in F$, then $c_T(\lambda) = \det(\lambda \operatorname{id} - T) = 0$. So $\lambda \operatorname{id} - T$ is not invertible.

Definition 1.4. The **trace** of a matrix $A = [a_{i,j}] \in M_n(R)$ is $\operatorname{tr}(A) = \sum_{i=1}^n a_{i,i}$.

$\operatorname{tr} : M_n(R) \to R$ is an additive homomorphism of R-modules.

Lemma 1.1. $c_A(a) = x^n - \operatorname{tr}(A)x^{n-1} + \cdots + (-1)^n \det(A)$.

Proof. To get the constant term, we have

$$c_A(0) = \det(-A) = (-1)^n \det(A).$$

To get the largest nonzero term, note that

$$\det(xI - A) = \sum_{\sigma \in S_n} (\operatorname{sign}(\sigma))(x\delta_{1,\sigma(1)} - a_{1,\sigma(1)}) \cdots (x\delta_{n,\sigma(n)} - a_{n,\sigma(n)}).$$

The coefficient of x^{n-1} comes form the term with $\sigma = \operatorname{id}$:

$$(x - a_{1,1}) \cdots (x - a_{n,n}) = x^n - (a_{1,1} + \cdots + a_{n,n})x^{n-1} + \cdots \qquad \square$$

Definition 1.5. If $Tv = \lambda v$ with $v \neq 0$, then $\lambda \in F$ is called an **eigenvalue** of T, and v is called an **eugenvector** for T. Then $E_\lambda(T) = \{v \in V : Tv = \lambda v\}$ is an F-subspace of V called the λ-**eigenspace** for T.

If $T : V \to V$ is an F-linear transofrmation, then V has an $F[x]$-module structure by $f(x) \cdot v := f(T)(v)$. We want to study the module structure. We might as well study the structure of finitely generated modules over PIDs.

1.3 Localization of modules

Let R be a commutative ring, let M be an R-module, and let S be a multiplicatively closed subset of R.

Lemma 1.2. *The relation $\sim_S$ on $S \times M$ defined by $(s, m) \sim_S (t, n)$ is there exists some $r \in S$ such that $r(sn - tm) = 0$ is an equivalence relation.*

Definition 1.6. The **localization** of M by S, called $S^{-1}M$ is the set of equivalence classes under $\sim_S$. We write m/s for the equivalence class of (s, m).

Lemma 1.3. *$S^{-1}M$ is an $S^{-1}R$-module under the operations*

$$\frac{m}{s} + \frac{n}{t} = \frac{tm + sn}{st}, \qquad \frac{r}{s} \cdot \frac{m}{t} = \frac{rm}{st}.$$

Example 1.1. Let $p \subseteq R$ be a prime ideal. Let $S_p = R \setminus p$. Then $R_p = S_p^{-1}R$. So $M_p = S_p^{-1}M$ is an R_p-module.

Example 1.2. Let $R = \mathbb{Z}$ and $M = \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z} \oplus \mathbb{Z}^2$. Then $M_{(3)} \cong \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}_{(3)}^2$, is a $\mathbb{Z}_{(3)}$-module, where $\mathbb{Z}_{(3)} = \{a/b : 3 \nmid b\}$.